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THE STABILITY OF THE STEADY MOTION OF A GYROSTAT WITH A LIQUID IN A CAVITY†

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A symmetrical rigid body with a spherical base, carrying a rotor and having a cavity in the shape of an ellipsoid of revolution, completely filled with an ideal incompressible liquid in uniform vortex motion, is moving along an absolutely rough plane. It is shown that this system admits of an energy integral, Jellett's integral, the integral of constant vorticity and a geometric integral. The construction of a Lyapunov function as a linear combination of first integrals [1] yields the sufficient conditions for the rotation of the gyrostat about the vertically positioned axis of symmetry to be stable. The conditions for the gyrostat's rotation to be unstable are found. It is shown that the rotor may prove to have either a stabilizing or destabilizing effect on the system and that the gyrostat admits of motions of the type of regular precession. The sufficient conditions for the stability of these motions are obtained.

The conditions for the regular precession of a symmetrical rigid body with a fixed point, having an ellipsoidal cavity completely filled with an ideal liquid, to be stable are well known [2]. The stability of rotations of a symmetrical rigid body, containing an ellipsoidal cavity completely filled with an ideal incompressible liquid, has been investigated in the case of motion on a smooth horizontal plane and a plane with sliding friction [3, 4]. For the case of an absolutely rough plane, the necessary condition for the rotations of a gyrostat to be stable have been found [3], and its oscillations about its equilibrium position have been investigated [5].

1. THE EQUATIONS OF MOTION. FIRST INTEGRALS

Consider the motion on an absolutely rough horizontal plane of a symmetrical gyrostat [6] – a heavy symmetrical rigid body with a spherical base, with a rotating symmetrical rotor whose axis is permanently attached to the body. The body contains a cavity in the shape of an ellipsoid of revolution, completely filled with a homogeneous ideal incompressible liquid in uniform vortex motion. It is assumed that the axis of symmetry of the body is also the axis of the rotor and of the cavity.

Let $O'xyz$ be a fixed right-handed system of coordinates with origin O' and x and y axes in the support plane, with the z axis pointing vertically upwards. We introduce a system of coordinates $G\xi_1\xi_2\xi_3$ rigidly attached to the gyrostat, with origin at its centre of mass G and axes pointing along its principal central axes of inertia, the ξ_3 axis being directed upwards along the axis of dynamic symmetry.

We shall assume that the geometrical centre C of the spherical base of the body is situated on the ξ_3 axis, denoting its coordinate along that axis by l and the radius of the spherical base by ρ.

The position of the gyrostat in the system $O'xyz$ is defined by the coordinates x_G and y_G of the centre of mass, the Euler angles ϑ, ψ and φ of the body and the rotor's angle of rotation δ relative to the body. The nutation angle ϑ is the angle between the ξ_3 axis and the vertical. We shall assume that ϑ, ψ and φ vary within the limits

$$0 \leqslant \vartheta \leqslant \pi/2, \quad 0 \leqslant \psi < 2\pi, \quad 0 \leqslant \varphi < 2\pi$$

Let γ denote the unit vector along the vertical. The coordinates of the radius vector $\mathbf{r}$ of the point O at which the body is in contact with the support plane are [7]

$$r_1 = -\rho\gamma_1, \quad r_2 = -\rho\gamma_2, \quad r_3 = l - \rho\gamma_3 \tag{1.1}$$

Let ω_i (i = 1, 2, 3) be the projections onto the ξ_1, ξ_2 and ξ_3 axes of the vector ω of the body's instantaneous angular velocity. We will assume that the generalized forces acting on the rotor vanish. The equations of motion of the rotor imply a first integral

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$$\dot{\delta}+\omega_3=\Omega_*=\text{const} \tag{1.2}$$

stating that the projection of the rotor's instantaneous absolute angular velocity onto its axis of rotation is constant [8].

The equations of the surface of the plane in terms of the ξ_1, ξ_2 and ξ_3 axes are

$$\xi_1^2/a_1^2+\xi_2^2/a_2^2+\left(\xi_3-\xi_3^0\right)^2/a_3^2=1 \tag{1.3}$$

where $a_1 = a_2$ and a_3 are the semi-axes of the cavity and ξ_3^0 is the coordinate of its geometric centre on the x_3 axis.

The Helmholtz equations of uniform vortex motion of the liquid in the cavity, in projections onto the ξ_1, ξ_2 and ξ_3 axes, are

$$\dot{\Omega}_1=2a_1^2\left(\frac{\omega_3\Omega_2}{a_1^2+a_2^2}-\frac{\omega_2\Omega_3}{a_3^2+a_1^2}\right)-2\frac{a_1^2\left(a_3^2-a_2^2\right)}{\left(a_1^2+a_2^2\right)\left(a_1^2+a_3^2\right)}\Omega_2\Omega_3 \quad (123) \tag{1.4}$$

where the symbol (123) means that the two unwritten equations are obtained from the written one by cyclic permutation of the subscripts 1, 2, 3; Ω_i (i = 1, 2, 3) are the projections onto the ξ_1, ξ_2 and ξ_3 axes of the vector (rot $\mathbf{v}_*$)/2, where $\mathbf{v}_*$ is the absolute velocity vector of the liquid particles.

The equations of motion of the gyrostat, referred to the ξ_1, ξ_2 and ξ_3 axes, are

$$m\left(\dot{v}_1+\omega_2v_3-\omega_3v_2\right)=-mg\gamma_1+R_1 \quad (123) \tag{1.5}$$

$$A_{*1}\dot{\omega}_1+A_1'\dot{\Omega}_1+\left(A_{*3}\omega_3+J\Omega_*+A_3'\Omega_3\right)\omega_2-\left(A_{*2}\omega_2+A_2'\Omega_2\right)\omega_3=M_1 \quad (123) \tag{1.6}$$

$$\dot{\gamma}_1=\omega_3\gamma_2-\omega_2\gamma_3 \quad (123) \tag{1.7}$$

$$v_1=r_2\omega_3-r_3\omega_2 \quad (123) \tag{1.8}$$

Equations (1.5) and (1.6) express the laws governing the variation of the momentum and angular momentum of the gyrostat, respectively, while Eqs (1.7) and (1.8) are, respectively, Poisson's equations and relations expressing the condition that the body is rolling on the plane without sliding. Here m is the mass of the gyrostat, with $m = m_1 + m_2$, where m_1 is the mass of the body–rotor system and m_2 is the mass of the liquid, v_i, R_i and M_i (i = 1, 2, 3) are the projections onto the ξ_1, ξ_2 and ξ_3 axes of the velocity vector of the centre of mass of the gyrostat, the reaction of the support plane and the moment of the reaction force about the point G, respectively, $A_{*i} = A_i + A_i^*$ (i = 1, 2, 3) are the moments of inertia of the transformed body [9], where A_1 and $A_2 = A_1$ are the moments of inertia of the body–rotor system about the ξ_1 and ξ_2 axes and A_3 and J are the moments of inertia of the body and the rotor, respectively, about the ξ_3 axis. The moments of inertia A_i^* of the equivalent rigid body [9] and the differences A_i' between the corresponding moments of inertia of the liquid and the equivalent rigid body are defined by the formulae

$$\begin{aligned}&A_1^*=\frac{1}{5}m_2\frac{\left(a_1^2-a_3^2\right)^2}{a_1^2+a_3^2}+m_2\left(\xi_3^0\right)^2, \quad A_3^*=0\\&A_1'=\frac{4}{5}m_2\frac{a_1^2a_3^2}{a_1^2+a_3^2}, \quad A_3'=\frac{2}{5}m_2a_1^2\end{aligned} \tag{1.9}$$

Equations (1.6) are identical with the equations of motion of a solid with moments of inertia A_{*i} (i = 1, 2, 3) and a rotor, attached to which is a rotating gyroscope with moments of inertia A_i' (i = 1, 2, 3), where the rotation of the gyroscope occurs, according to Eqs (1.4), in such a way that the geometry of the masses of the system remains unchanged. Consequently, the effect of the liquid, which is performing uniform vortex motion, is identical to the effect of a certain equivalent body and a rotating gyroscope, which are attached to the body–rotor system [1].

Having determined the quantities R_i (i = 1, 2, 3) from Eqs (1.5) and (1.8), we find

$$M_1 = m\left[-gl\gamma_2 - \left(r_2^2 + r_3^2\right)\dot{\omega}_1 + r_1\left(r_2\dot{\omega}_2 + r_3\dot{\omega}_3\right) + lr_2\left(\omega_1^2 + \omega_2^2\right) + lr_3\omega_2\omega_3\right]$$

$$M_2 = m\left[gl\gamma_1 - \left(r_1^2 + r_3^2\right)\dot{\omega}_2 + r_2\left(r_1\dot{\omega}_1 + r_3\dot{\omega}_3\right) - lr_1\left(\omega_1^2 + \omega_2^2\right) - lr_3\omega_1\omega_3\right] \quad (1.10)$$

$$M_3 = m\left[-\left(r_1^2 + r_2^2\right)\dot{\omega}_3 + r_3\left(r_1\dot{\omega}_1 + r_2\dot{\omega}_2\right) + l\left(r_2\omega_1 - r_1\omega_2\right)\omega_3\right]$$

Taking Eqs (1.10) into consideration, Eqs (1.4), (1.6) and (1.7) form a complete system of nine differential equations of motion for the gyrostat.

Note that the following relation is obtained immediately from the third Helmholtz equation (1.4) and from equalities (1.9)

$$A_3'\dot{\Omega}_3 = A_1'\left(\Omega_1\omega_2 - \Omega_2\omega_1\right)$$

In view of this equality, the third equation in system (1.6) may be written in the form

$$d\left(A_{*3}\omega_3 + J\Omega_*\right)/dt = M_3$$

The system of equations of motion of the gyrostat and the liquid in its cavity admits of several first integrals:

– the energy integral

$$\begin{aligned} U_0 = {} & \left[A_{*1} + m\left(r_2^2 + r_3^2\right)\right]\omega_1^2 + \left[A_{*1} + m\left(r_1^2 + r_3^2\right)\right]\omega_2^2 + \left[A_{*3} + m\left(r_1^2 + r_2^2\right)\right]\omega_3^2 - \\ & -2mr_1r_2\omega_1\omega_2 - 2mr_1r_3\omega_1\omega_3 - 2mr_2r_3\omega_2\omega_3 + \\ & +A_1'\left(\Omega_1^2 + \Omega_2^2\right) + A_3'\Omega_3^2 - 2mgl\gamma_3 = c_0 = \text{const} \end{aligned} \quad (1.11)$$

– the generalized Jellett integral

$$\begin{aligned} U_1 = {} & A_{*1}\left(r_1\omega_1 + r_2\omega_2\right) + r_3\left(A_{*3}\omega_3 + J\Omega_*\right) + \\ & +A_1'\left(r_1\Omega_1 + r_2\Omega_2\right) + \left(r_3 - l\right)A_3'\Omega_3 = c_1 = \text{const} \end{aligned} \quad (1.12)$$

– the integral of constant vorticity

$$U_2 = a_3^2\left(\Omega_1^2 + \Omega_2^2\right) + a_1^2\Omega_3^2 = c_2 = \text{const} \quad (1.13)$$

– the geometrical integral

$$U_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \quad (1.14)$$

Relation (1.12) may be written in the form

$$U_1 = \left(\mathbf{K}_*, \mathbf{r}\right) - \rho\left(\mathbf{K}', \boldsymbol{\gamma}\right) = c_1 = \text{const}$$

This integral states that the difference between two scalar products is a constant. The term $(\mathbf{K}_*, \mathbf{r})$ is the scalar product of the angular momentum vector $\mathbf{K}_*$ of the transformed body–rotor system for the point G and the radius vector of the point at which the body is in contact with the support plane. The term $\rho(\mathbf{K}', \boldsymbol{\gamma})$ is the product of the radius of the spherical base with the scalar product of the vector $\mathbf{K}'$ of the momentum of a rotating gyroscope with moments of inertia A_1', A_3' [1] and the unit vector of the vertical.

Jellett's integral was first obtained for a single rigid body rolling over a rough plane [10, 11]. It was later shown that Jellett's integral exists in the case of a body with a rotor [8] and an extension was obtained for a liquid-filled body on a plane with friction [4]. We have just extended this integral to the case of a liquid-filled gyroscope on an absolutely rough surface.

2. STEADY MOTIONS. STABILITY

Motion about a vertically directed axis of symmetry. The equations of motion of the gyrostat have a particular solution

$$\begin{aligned}&\omega_1=\omega_2=0,\quad \omega_3=\omega_3^0,\quad \Omega_1=\Omega_2=0,\quad \Omega_3=\Omega_3^0\\ &\gamma_1=\gamma_2=0,\quad \gamma_3=1\end{aligned}\tag{2.1}$$

which describes uniform rotation of the gyrostat about a vertically directed axis of symmetry and relative elliptical rotation of the liquid about the same axis. Let us take this as the unperturbed motion and analyse its stability relative to the variables ω_i, Ω_i and γ_i (i = 1, 2, 3).

In perturbed motion we put

$$\omega_3=\omega_3^0+x,\quad \gamma_3=1+y,\quad \Omega_3=\Omega_3^0+z\tag{2.2}$$

The previous notation is retained for the other variables.

To construct the Lyapunov function, we use Chetayev's method [12, 1]. Consider the function

$$\begin{aligned}V={}&U_0+2\frac{\omega_3^0}{\rho(1-\varepsilon)}U_1+\frac{A_3'}{a_1^2}\left[\frac{\omega_3^0}{(1-\varepsilon)\Omega_3^0}-1\right]U_2+\left[mgl+\frac{\omega_3^0}{1-\varepsilon}\left(A_{*3}\omega_3^0+J\Omega_*+A_3'\Omega_3^0\right)\right]U_3=\\ &=\tilde{A}\left(\omega_1^2+\omega_2^2\right)+A_{*3}x^2+m\rho^2\left(\omega_3^0\right)^2\left(\gamma_1^2+\gamma_2^2\right)-\\ &-2m\rho(\rho-l)\omega_3^0(\omega_1\gamma_1+\omega_2\gamma_2)+A_1'\left(\Omega_1^2+\Omega_2^2\right)+A_3'z^2-\\ &-2\frac{\omega_3^0}{1-\varepsilon}\left[A_{*1}(\omega_1\gamma_1+\omega_2\gamma_2)+(A_{*3}x+A_3'z)y+A_1'(\Omega_1\gamma_1+\Omega_2\gamma_2)\right]+\\ &+\frac{A_3'}{a_1^2}\left[\frac{\omega_3^0}{(1-\varepsilon)\Omega_3^0}-1\right]\left[a_3^2\left(\Omega_1^2+\Omega_2^2\right)+a_1^2z^2\right]+\\ &+\left[mgl+\frac{\omega_3^0}{1-\varepsilon}\left(A_{*3}\omega_3^0+J\Omega_*+A_3'\Omega_3^0\right)\right]\left(\gamma_1^2+\gamma_2^2+y^2\right)\end{aligned}$$

where

$$\varepsilon=l/\rho,\quad \tilde{A}=A_{*1}+m(\rho-l)^2$$

The function V is the sum of three quadratic forms, two of which have the same matrix. Applying Sylvester's criterion, we find the conditions for the function V to be positive-definite

$$\begin{aligned}&\frac{1}{p_1(1-\varepsilon)}-\tilde{\alpha}>0,\quad p_1=\frac{\Omega_3^0}{\omega_3^0},\quad \tilde{\alpha}=\frac{\alpha-1}{\alpha+1},\quad \alpha=\frac{a_3^2}{a_1^2}\\ &\left[\frac{1}{p_1(1-\varepsilon)}-\tilde{\alpha}\right]\left\{\frac{mgl}{\left(\omega_3^0\right)^2}+\frac{1}{(1-\varepsilon)^2}\left[(A_{*3}+Jp_2)(1-\varepsilon)-A_{*1}\right]\right\}+\\ &+A_3'\frac{\tilde{\alpha}}{1-\varepsilon}\left(\frac{\tilde{\alpha}}{1-\varepsilon}-p_1\right)>0,\quad p_2=\frac{\Omega_*}{\omega_3^0}\\ &\frac{mgl}{\left(\omega_3^0\right)^2}+\frac{p_1}{(1-\varepsilon)^2}\left[(A_3'+Jp_3)(1-\varepsilon)-\varepsilon A_{*3}\right]>0,\quad p_3=\frac{\Omega_*}{\Omega_3^0}\\ &\frac{1}{p_1(1-\varepsilon)}\left\{\frac{mgl}{\left(\omega_3^0\right)^2}+\frac{1}{(1-\varepsilon)^2}\left[Jp_2(1-\varepsilon)-\varepsilon A_{*3}\right]\right\}>0\end{aligned}\tag{2.3}$$

By Lyapunov's stability theorem [12, 1], inequalities (2.3) are the sufficient conditions for the unperturbed motion (2.1) to be stable relative to the variables ω_i, Ω_i and γ_i ($i = 1, 2, 3$).

In the particular motion (2.1) the quantities ω_3^0 and Ω_3^0 may take arbitrary values. The values in the interval $0 < \tau \leqslant (\Omega_3^0/\omega_3^0) \leqslant 1$, where τ is an infinitesimal quantity, are of practical interest, since, in uniform rotation of the gyrostat about a vertically positioned axis of symmetry, the liquid, if it is not previously in vortex motion, first performs vortex-free motion and is then gradually drawn into the motion of the body until it is moving together with it as a single solid body [13].

Put $\omega_3^0 = \Omega_3^0$. Then stability conditions (2.3) become

$$
\begin{gathered}
\frac{1}{1-\varepsilon}-\tilde{\alpha}>0 \\
\left(\frac{1}{1-\varepsilon}-\tilde{\alpha}\right)\left\{\frac{mgl}{\left(\omega_3^0\right)^2}+\frac{1}{(1-\varepsilon)^2}\left[\left(A_{*3}+Jp_2\right)(1-\varepsilon)-A_{*1}\right]\right\}+A_3'\frac{\tilde{\alpha}}{1-\varepsilon}\left(\frac{\tilde{\alpha}}{1-\varepsilon}-1\right)>0 \\
\frac{mgl}{\left(\omega_3^0\right)^2}+\frac{1}{(1-\varepsilon)^2}\left[\left(A_3'+Jp_2\right)(1-\varepsilon)-\varepsilon A_{*3}\right]>0 \\
\frac{1}{1-\varepsilon}\left\{\frac{mgl}{(\omega_3^0)^2}+\frac{1}{(1-\varepsilon)^2}[Jp_2(1-\varepsilon)-\varepsilon A_{*3}]\right\}>0
\end{gathered}
\tag{2.4}
$$

Note that always

$$1/(1-\varepsilon)>0$$

Indeed, if the gyrostat's centre of mass G lies above the centre C of the spherical base, then $l < 0$ and so $1-\varepsilon > 0$; if G is below C, then $0 < l < \rho$ and so $1-\varepsilon > 0$. Hence, if the last inequality of (2.4) holds, the penultimate one also holds.

It is obvious from conditions (2.4) that if the body–liquid system and the rotor are rotating in the same sense, and the cavity has the shape of an oblate ellipsoid ($\alpha < 1$), this will have a stabilizing effect. But if the rotor and the body are rotating in opposite senses and the angular velocity Ω_* of the rotor significantly exceeds the magnitude ω_3^0 of the body's angular velocity, the rotor will have a destabilizing effect on the system. A cavity in the form of a prolate ellipsoid ($\alpha > 1$) is also a destabilizing factor.

Suppose the centre of mass of the gyrostat lies above the centre of the spherical base ($l < 0$). If the cavity is a strongly prolate ellipsoid ($\alpha > 1 + 2\rho/|l|$), the first condition of (2.4) is violated.

If $\Omega_3^0/\omega_3^0 = \tau \ll 1$, when the motion of the liquid in the cavity is very close to potential motion, conditions (2.3) become

$$
\begin{gathered}
\frac{mgl}{(\omega_3^0)^2}+\frac{1}{(1-\varepsilon)^2}[(A_{*3}+Jp_2)(1-\varepsilon)-A_{*1}]>0 \\
\frac{mgl}{(\omega_3^0)^2}+\frac{1}{(1-\varepsilon)^2}[Jp_2(1-\varepsilon)-\varepsilon A_{*3}]>0
\end{gathered}
\tag{2.5}
$$

The first of these conditions, obtained for a gyrostat moving on a plane of arbitrary roughness, also holds in the case of viscous and dry Coulomb friction [7, 14].

In the special case of a *spherical cavity* ($\alpha = 1$), the first of conditions (2.3) means that the liquid and the body rotate in the same direction. In that case the rotor cannot stabilize the system. If there is no rotor, this result is analogous to the case of a top with a cavity on a plane with sliding friction [4]. If the body and the liquid in the cavity are also rotating at the same angular velocity, one obtains inequalities (2.5).

The linearized equations of perturbed motion are

$$
\begin{gathered}
\dot{\omega}_1=-\Gamma_1\omega_2+\Gamma_2\Omega_2-\Gamma_3\gamma_2, \quad \dot{\omega}_2=\Gamma_1\omega_1-\Gamma_2\Omega_1+\Gamma_3\gamma_1, \quad \dot{x}=0 \\
\dot{\Omega}_1=-\Gamma_4\omega_2+\Gamma_5\Omega_2, \quad \dot{\Omega}_2=\Gamma_4\omega_1-\Gamma_5\Omega_1, \quad \dot{z}=0 \\
\dot{\gamma}_1=-\omega_2+\omega_3^0\gamma_2, \quad \dot{\gamma}_2=\omega_1-\omega_3^0\gamma_1, \quad \dot{y}=0
\end{gathered}
\tag{2.6}
$$

where

$$\Gamma_1 = \tilde{B}\omega_3^0 + \tilde{J}\Omega_* + \tilde{A}_3'\tilde{\alpha}^2\Omega_3^0, \quad \Gamma_2 = 2\tilde{A}_3'\frac{\alpha\tilde{\alpha}}{\alpha+1}\Omega_3^0,$$

$$\Gamma_3 = \frac{mgl}{\tilde{A}}, \quad \Gamma_4 = \frac{2}{\alpha+1}\Omega_3^0, \quad \Gamma_5 = \omega_3^0 - \tilde{\alpha}\Omega_3^0 \tag{2.7}$$

$$\tilde{B} = \frac{A_{*3} - A_{*1} + m(\rho - l)l}{\tilde{A}}, \quad \tilde{J} = \frac{J}{\tilde{A}}, \quad \tilde{A}_3' = \frac{A_3'}{\tilde{A}}$$

The characteristic equation of system (2.6) is

$$\Delta(\varkappa) = \varkappa^3(\varkappa^6 + \Lambda_1\varkappa^4 + \Lambda_2\varkappa^2 + \Lambda_3) = 0 \tag{2.8}$$

where

$$\Lambda_1 = (2+\tilde{B}^2)(\omega_3^0)^2 + \tilde{J}^2\Omega_*^2 + 2\tilde{J}(\tilde{B}\omega_3^0 + \tilde{A}_3'\tilde{\alpha}^2\Omega_3^0)\Omega_* +$$
$$+ 2\tilde{\alpha}(\tilde{B}\tilde{A}_3'\tilde{\alpha} - 1)\omega_3^0\Omega_3^0 + \tilde{\alpha}^2\left[(\tilde{A}_3')^2\tilde{\alpha} + 1 - 8\tilde{A}_3'\frac{\alpha}{\alpha^2-1}\right](\Omega_3^0)^2 + 2\Gamma_3$$

$$\Lambda_2 = (1+2\tilde{B}^2)(\omega_3^0)^4 + \tilde{J}[2\tilde{B}(\omega_3^0)^3 + \tilde{A}_3'\tilde{\alpha}^2(\Omega_3^0)^3]\Omega_* +$$
$$+4\tilde{J}\tilde{\alpha}(\tilde{A}_3'\tilde{\alpha} - \tilde{B})(\omega_3^0)^2\Omega_*\Omega_3^0 - 2\tilde{J}\tilde{\alpha}\left[2\tilde{A}_3'\frac{\alpha^2+1}{(\alpha+1)^2} - \tilde{B}\tilde{\alpha}\right]\omega_3^0\Omega_*(\Omega_3^0)^2 +$$
$$+\tilde{J}^2[2(\omega_3^0)^2 - 2\tilde{\alpha}\omega_3^0\Omega_3^0 + \tilde{\alpha}^2(\Omega_3^0)^2]\Omega_*^2 + 2\tilde{\alpha}(2\tilde{B}\tilde{A}_3'\tilde{\alpha} - 1 - \tilde{B}^2)(\omega_3^0)^3\Omega_3^0 +$$
$$+2\tilde{A}_3'\tilde{\alpha}^2(\tilde{B} + \tilde{A}_3'\tilde{\alpha})\omega_3^0(\Omega_3^0)^3 + \tilde{\alpha}\left[2(\tilde{A}_3')^2\tilde{\alpha}^3 + \left(1 - 8\tilde{A}_3'\frac{\alpha}{\alpha^2-1} + \tilde{B}^2\right)\tilde{\alpha} -\right.$$
$$\left. -4\tilde{B}\tilde{A}_3'\frac{\alpha^2+1}{(\alpha+1)^2}\right](\omega_3^0)^2(\Omega_3^0)^2 + \Gamma_3^2 + 2\Gamma_3\left[(1+\tilde{B})(\omega_3^0)^2 + \tilde{J}\omega_3^0\Omega_* +\right.$$
$$\left. +\tilde{\alpha}(\tilde{A}_3'\tilde{\alpha} - 2)\omega_3^0\Omega_3^0 + \tilde{\alpha}\left(\tilde{\alpha} - \tilde{A}_3'\frac{4\alpha}{(\alpha+1)^2}\right)(\Omega_3^0)^2\right]$$

$$\Lambda_3 = \{(\omega_3^0 - \tilde{\alpha}\Omega_3^0)[\tilde{B}(\omega_3^0)^2 + \tilde{J}\omega_3^0\Omega_* + \Gamma_3] + \tilde{A}_3'\tilde{\alpha}(\tilde{\alpha}\omega_3^0 - \Omega_3^0)\omega_3^0\Omega_3^0\}^2$$

The characteristic equation (2.8) has three zero roots and six non-zero roots, as determined from the equation

$$\varkappa^6 + \Lambda_1\varkappa^4 + \Lambda_2\varkappa^3 + \Lambda_3 = 0$$

which contains $\varkappa$ in even powers only. A necessary condition for the motion (2.1) to be stable is that all the roots of this equation must be imaginary. This means that the squared roots $\varkappa^2$ must be real and negative. This condition can be satisfied by requiring that the coefficients Λ_i (i = 1, 1, 3) satisfy the Hurwitz condition [12]

$$\Lambda_1 > 0, \quad \Lambda_1\Lambda_2 - \Lambda_3 > 0 \tag{2.9}$$

and the condition for the roots of Eq. (2.8), as a cubic equation in $\varkappa^2$, to be real [15]

$$\Lambda_2^2(4\Lambda_2 - \Lambda_1^2) + 27\Lambda_3^2 + 2\Lambda_1\Lambda_3(2\Lambda_1^2 - 9\Lambda_2) < 0 \tag{2.10}$$

Thus, inequalities (2.9) and (2.10) are the necessary conditions for the motion (2.1) to be stable. If there is no rotor ($J\Omega_* = 0$) and also $\Omega_3^0 = \omega_3^0$, the necessary stability condition was obtained previously [3].

If inequality (2.10) is satisfied but at least one of conditions (2.9) fails to hold, some of the roots of Eq. (2.8) will be real and positive. In that case the unperturbed motion (2.1) is unstable [12].

In the case of a *spherical cavity*, inequality (2.10) reduces to the form

$$\{[A_{*3}+m\rho(\rho-l)]\omega_3^0+J\Omega_*\}^2+4[A_{*1}+m(\rho-l)^2]mgl>0 \tag{2.11}$$

This is known to be the condition for the stability of a top with a rotor on an absolutely rough plane [16]. Inequalities (2.9) become

$$2(\omega_3^0)^2+[(\tilde{C}-1)\omega_3^0+\tilde{J}\Omega_*]^2+2\Gamma_3>0, \quad \tilde{C}=[A_{*3}+m\rho(\rho-l)]/\tilde{A}$$

$$\{(\omega_3^0)^2+[(\tilde{C}-1)\omega_3^0+\tilde{J}\Omega_*]^2+2\Gamma_3\}\times$$

$$\times\{[\tilde{C}(\omega_3^0)^2+\tilde{J}\omega_3^0\Omega_*+\Gamma_3]^2+(\tilde{C}\omega_3^0-2\omega_3^0+\tilde{J}\Omega_*)^2(\omega_3^0)^2\}>0$$

Thus, the condition for the instability of a top with a rotor is the combination of inequalities (2.11) and the following inequality

$$\tilde{A}^2(\omega_3^0)^2+[(\hat{C}-\tilde{A})\omega_3^0+J\Omega_*]^2+2\tilde{A}mgl<0, \quad \hat{C}=\tilde{C}\tilde{A}$$

Motion of the type of regular precession. The equations of motion of the gyrostat have a solution

$$v_i=0, \quad \omega_i^0=\lambda(\gamma_i^0-\varepsilon\delta_{i3}) \tag{2.12}$$

$$\Omega_i^0=\lambda\Theta_i\gamma_i^0, \quad \Theta_{1,2}=\frac{A_1'}{A_1'+\mu a_3^2}, \quad \Theta_3=\frac{A_3'}{A_3'+\mu a_1^2}, \quad i=1,2,3$$

where λ and μ are arbitrary constants, and γ_3^0 is a solution of the equation

$$\lambda^2[A_{*1}+A_1'\Theta_1-A_{*3}(\gamma_3^0-\varepsilon)-A_3'\Theta_3]\gamma_3^0-\lambda J\Omega_*=mgl$$

Solution (2.12) describes the motion of a gyrostat of the type of regular precession, with the velocity of the centre of mass G equal to zero and the angular velocities of precession $\dot\psi_0$ and of spin $\dot\varphi_0$ equal to λ and $-\lambda\varepsilon$, respectively. The point O of contact of the body with the support plane describes a circle of radius $|l|\sin\vartheta_0$ [17].

Let us take (2.12) as the unperturbed motion and analyse its stability. The Lyapunov function will be a linear combination of the first integrals (1.11)–(1.14)

$$V=U_0+2\frac{\lambda}{\rho}U_1+\mu U_2+\lambda^2(A_{*1}+A_1'\Theta_1)U_3=$$

$$=m\mathbf{v}^2+A_{*1}(x_1^2+x_2^2)+(A_1'+\mu a_3^2)(z_1^2+z_2^2)+A_{*3}\omega_3^2+(A_3'+\mu a_1^2)\Omega_3^2-$$

$$-2mgl\gamma_3-2\lambda(A_{*3}\omega_3+J\Omega_*)(\varepsilon-\gamma_3)-2\lambda A_3'\Omega_3\gamma_3+\lambda^2(A_{*1}+A_1'\Theta_1)\gamma_3^2$$

where

$$x_j=\omega_j^0-\lambda\gamma_j^0, \quad z_j=\Omega_j^0-\lambda\Theta_j\gamma_j^0, \quad j=1,2$$

In the perturbed motion, put

$$\omega_3=\omega_3^0+x_3, \quad \gamma_3=\gamma_3^0+y_1, \quad \Omega_3=\Omega_3^0+z_3$$

retaining the previous notation for the other variables. Then the Lyapunov function V is a positive-definite quadratic form in the variables v_i, x_i, y_1 and z_i ($i=1,2,3$), provided that the following equalities hold:

$$A_1'+\mu a_3^2>0, \quad A_{*1}+A_1'\Theta_1-A_{*3}>0$$

$$\frac{A_{*3}}{A_3'+\mu a_1^2}[A_{*1}+A_1'\Theta_1-A_{*3}-A_3'\Theta_3]>0 \tag{2.13}$$

These equalities are the sufficient conditions for the stability of the unperturbed motion (2.12) relative to the variables v_i, x_j, z_j, γ_3, ω_3 and Ω_3 ($i = 1, 2, 3; j = 1, 2$). It follows from these inequalities that in the case of a gyrostat on an absolutely rough plane, the sufficient condition for the stability of the motion (2.12) is narrower than in the case of a rigid body with a fixed point and an ellipsoidal cavity completely filled with an ideal liquid [2].

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